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# EXAMINING THE EXAMPLE GENERATION ABILITIES OF HIGH SCHOOL STUDENTS WITHIN THE CONTEXT OF MATHEMATICS COURSE

*Case Study*

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**Abstract**

Examples are an indispensable part of mathematical thinking through analogy, learning and the teaching process. In this study, the strategies that high school students used in example generation activities in mathematics course; operational and conceptual knowledge levels were determined and their effect on example generation ability was examined. The working group of the study was designed as a case study that followed a qualitative research method; the study consisted of 22 students attending high school in the 2016-2017 academic year. As a data collection tool, example generation questions were administered to the students. To analyze students' example generation processes in depth, semi-structured interviews were conducted and data obtained from the interviews were analyzed using a content analysis method. The results showed that the students used trial and error strategy more than transformation strategy while generating examples but no finding of analysis strategy was obtained. Factors affecting the students' example generation skills negatively included the lack of a mathematical equivalent of the concept images related to the topics covered by the questions and their not having a high enough level of concept knowledge. Results also indicated that conceptual knowledge had a positive effect on the ability to generate examples.

***Keywords:*** example generation, conceptual knowledge, procedural knowledge

## 1. Introduction

The purpose of a secondary education mathematics program is to raise students to be good problem solvers who have developed mathematical thinking skills. This program emphasizes mathematical concepts, relations between them, basic mathematical operations, and their mathematical meanings (TTKB, 2017). However, students sometimes do not understand the mathematical thinking behind operations, rules, and formulas (Hiebert & Lefever, 1986). For this reason, in order to understand the mathematical concepts and relations between these concepts, the activities that involve questions to use and learn conceptual knowledge need to be implemented in the classroom environment. However, it can be said that such activities do not take place much in the secondary school mathematics program. Although mathematics teachers often want to use these activities, they do not apply these activities to class because of concerns about the completion of the mathematics program topics on time according to the annual course schedule. With this study, it is aimed to contribute to activities that are not encountered very much in secondary school mathematics lessons and to teachers who want to use these activities in lessons.

Examples have an important role in learning mathematical concepts, techniques and reasoning, and in the development of mathematical competence (Bills, Dreyfus, Mason, Tsamir, Watson & Zaslavsky, 2006). Examples vary according to the situations they are used in. Examples with strong visual characteristics and specific features are called prototype examples (Hershkowitz, 1990). For example, the number $\sqrt{2}$, used for irrational numbers, emerges as a prototype. Beyond prototypes, in order to go beyond the boundaries as the definition allows and to be aware of the boundaries in this process, it is also important to use extraordinary examples (Bills et al., 2006). Examples that can be used by teacher or student in different situations related to the topic and that help to eliminate and clarify some of the mathematical ambiguities are defined as "reference" examples. An example of this is the use of $R^2$ to understand how the situations in the real analysis are formed (Michener, 1978). Examples needing a counter hypothesis or claim, in the context of a concept, procedure or proof, are called counter examples (Bills et al., 2006). The function f (x) = | x | is a counter example for the proposition "a continuous function is differentiable."

Example generation is a problem-solving activity in which individuals can develop different strategies (Zaslavsky & Peled, 1996). This activity emerges as some kind of open-ended problem in which the subject examines whether the wanted example exists, and shows why if it does not (Antonini, 2006). In this process, students make sense of the concepts by generating examples as learning strategies (Dahlberg & Housman, 1997). In order to learn a concept, it is necessary to construct rich examples related to that concept (Tall & Vinner, 1981**).** In the creation of conceptual learning, example generation activities have a very strong potential, in the pedagogical sense (Watson & Mason, 2005). In the study of Wagner, Orme, Turner and Yopp (2017) a teaching experiment involving example generation tasks was conducted for 8 weeks with 42 university students who had not previously taken a first-semester calculus course. In these activities, students were asked to explain their ideas about mathematical concepts novel for them and to generate examples. As a result of the study, it was seen that the students were able to explain the mathematical concepts that became more and more complex provided that the students sufficiently engaged in the tasks and their example generation abilities have improved in this way.

The way how example generation is used, the situations in which it is successful, the factors affecting the example generation process, and the way how the students develop strategies when generating examples are among the topics investigated in this context. Antonini (2006) identified the strategies that mathematicians use during example generation activities, based on the definition of example generation, and collected them under three headings:

- *Trial and Error: The example is sought among some recalled objects; for each example, the individual observes in most cases, whether the requested properties exist or not.*
- *Transformation: A number of transformations are made on the example that the individual has written to provide one or more requested properties, until all other requested properties are met.*
- *Analysis: It is assumed that the object constructed by individual provides some features which added to restrict and simplify the search ground. Other features occur with processes that remind a known concept or method to generate the requested example.*

Antonini (2006) states that expert mathematicians sometimes use all the example generation strategies together and they switch in between these strategies. Also, the transition from trial and error to analysis requires higher level thinking skills. Iannone, Inglis, Mejia-Ramos, Siemons and Weber (2009) stated that novice mathematicians (university students) used trial and error quite often, while analysis was very rarely used in this process, and students were quite inadequate in switching between the strategies.

Most of the students think that mathematics learning is to operate on some symbols, and that mathematics is learned by memorization (Soylu & Aydın, 2006). Procedural knowledge is in the foreground with these students, who have the ability to apply the rules without understanding their reasons (Skemp, 1971). Procedural knowledge is considered as both the symbolic language of mathematics and the knowledge of the rules and procedures used to solve problems (Hiebert & Lefever, 1986). The type of information in which the algorithms take place and process steps used in problem solving can also be treated as procedural knowledge (Star, 2007). However, a student with conceptual learning is a problem solver that generates mathematical knowledge using creativity and intuition (Bell & Baki, 1997). In the literature, conceptual understanding can be used instead of conceptual knowledge (Anderson, 2000; Rittle- Johnson & Schneider, 2015). Conceptual knowledge is the ability to understand which operation is to be done and why. When newly learned information is associated in accordance with the old knowledge, learning about the concept in question occurs (Skemp, 1971).

At national evaluation activity for educational development (Post, 1981), students were asked, "Which one of the numbers 1,2, 19 or 21 will be close to the result when 7/8 is added to 12/13?" Only 23% of children aged 13 years in the country gave the right answer to this question. More than half (55%) thought the answer was 19 or 21. Such answers show that the concept is misunderstood. A correct and conceptually based approach will "7/8 and 12/13 about 1, so their sum is about 2" form. Instead of using this idea based on conceptual knowledge, most of the students have tried a procedure (possibly from memory) to find a solution. This can be given as an example for conceptual and procedural knowledge. However, conceptual and procedural knowledge can be included in each other and it may not be the case that they are completely separated (Carpenter, 1986). Both types of knowledge are very important for mathematics education and their contributions to the individual's mental development are rather high (Post & Cramer, 1989).

In this process, students' understanding and ways of thinking are displayed by asking students to generate examples related to problems and objects (Hazzan & Zazkis, 1999; Van den Heuvel-Panhuizen, 1995; Watson & Mason, 2005; Zaslavsky, 1997), it is inevitable to encounter concept images. The concept image contains what arouses in the mind related to that concept. Therefore, when example generation efficiency is used as an evaluation tool by teachers, it can be determined whether there are any misconceptions about any mathematical concepts. Thus, factors that affect students' example generation skills may arise. On the other hand students' concept images and therefore their conceptual and procedural knowledge generally emerge as a result of their experience with examples and non-examples of a concept (Vinner & Dreyfus, 1989). When defining a concept, students who try to generate an example place more examples on the image for a mathematical concept and they use these examples better than students using other learning strategies (Dahlberg & Housman, 1997). For this reason, more example generation activities should be designed by the teachers in order to prevent the factors that negatively affect the example generation skills of the students.

Example generation activities, which are used by teachers as evaluation and teaching strategies, and by students for development of reasoning, conceptual learning, and mathematical competence, are thought to be an important subject to be analyzed in mathematics teaching. In the literature, the participants of the studies about the examples are generally teachers (Ng & Dindyal, 2015; Zaslavsky & Peled, 1996; Zodik & Zaslavsky, 2008) and university students (Antonini, 2006; Sağlam & Dost, 2015; Dahlberg & Housman, 1997; Edwards & Alcock, 2010; Iannone et al., 2009). There is a need for studies specifically regarding high school students' ability to generate examples within the context of a mathematics course (Zaslavsky & Ron, 1998). This study is thought to contribute to fill in this gap in the literature.

The aim of this research is to evaluate the example generation skills of high school students in the context of the strategies they use, concept image, and conceptual and procedural knowledge, so that the factors affecting the example generation process can be determined.

## 2. Method

This research is designed as a case study, which is a qualitative research method. In the case study, the information contained in a limited system is described and examined in depth, so the reader understands the product better (Merriam, 2013). In this study, the effects of concept images, conceptual and procedural knowledge, and the strategies used by high school students in the process of example generation were investigated.

### 2.1. Participants

Participants of the research were students attending an Anatolian High School in Nevşehir in the 2016-2017 academic year. First, questionnaires on example generation were administered to 129 students randomly selected from all grade levels of the high school. The questionnaire responses, belonging to the students, were then examined by each researcher independently. Based on the data obtained, semi-structured interviews were conducted with 22 students, who were expected to provide rich and in-depth data to the research, according to the purposeful sampling strategy. In the selection of these 22 students, volunteer participation was used as the basis and the following criteria were taken into consideration:

(i) Attempting to solve most of the questions used in the research

(ii) Academic achievement in mathematics courses

(iii) Ability to express their opinions in the direction of their teachers' suggestions and to defend their solutions.

Table 1: *Number of Students Participating in the Interview According to Grade Level*

| | | |
|---|---|---|
| Grade | 9th Grade | 9 |
| | 10th Grade | 3 |
| | 11th Grade | 5 |
| | 12th Grade | 5 |

### 2.2. Data collection process

The data collected in the study were obtained from two sources. These consisted of written documents containing the answers of students to example generation questions, and semi-structured interview records and their analysis.

In the preparation stage of the example generation questions, first the topics in the curriculum of the secondary school mathematics course were determined and questions were created according to the topics. In addition, it was decided to prepare questions that were not much included in mathematics courses and textbooks in order to investigate the effect of the students' conceptual knowledge level on the example generation ability. In this phase, the first writer who is high school teacher has been consulted about the suitability of the questions created in the mathematics curriculum, the relevance of the topics in the recent past, and the stated quality (not included in the textbooks too much). In addition, in order to contribute to the reliability and validity of the questions, the views of an academician working in the field of mathematics education were taken. Then a pilot study was done. As a result, it was identified that some of the questions on the form would not reveal the example generation skills of the students, and these questions were not used. For example, the question "give examples of two functions that cannot be composed" was omitted.

In the last stage, the final version of the example generation questions was established by taking expert opinions again.

1.2.1 Questions used in research

In this section, there are questions used in the research. It was emphasized that the questions used in the research are aimed at revealing the level of conceptual knowledge of the students. In addition by introducing some restrictions on the questions students were expected to think differently from the prototypes and customary examples they encounter in classroom and course books.

Research questions included questions about functions and equations that were common to all students at each grade level (9th, 10th, 11th, and 12th grade), and a question about the subject limit and continuity in 12th grade curriculum for 12th grade students. The questions used in the research are listed below.

1) Give a function example defined from Z to N, with a set of Z integers and a set of N natural numbers.

2) Give an example by drawing a graph for an expression that does not specify a function.

3) "Child-mother" mapping specifies a function with domain “children” and codomain “their mothers.” Give another function example from everyday life.

4) Give a function example in which "m is the function of n" when m and n are variables.

5) Give an example of a first-degree equation, in one variable, in which the solution set is a null set.

6) Give a function example whose limit exists at $x_0 \epsilon R$ but discontinuous at this point.

Questionnaires on example generation have been made in one course hour (40 minutes) for each level under the supervision of the mathematics teacher and the researchers. Appointments were scheduled with the students where semi-structured interviews were conducted in a suitable environment at school outside the class hours. Each interview, which was conducted individually, took 20-25 minutes on average and a voice recording was taken with the permission of the students. During the interviews, which started four days after the practice, students are asked to explain the solutions they have made and to think aloud.

### 2.3. Analysis of data

Content analysis was used to analyze the data collected. The main purpose of content analysis is to create concepts that can explain collected data, to make meaningful relations between the concepts and to provide relevant explanations (Yıldırım & Şimşek, 2011). In the first phase of the analysis, the students' written responses to the questions were examined in order to obtain information about the nature of the generated examples. This provided information about diversity of the examples and the strategies students used while generating examples.

Later, interviews with students were transferred to written text and read carefully by each author. In the analysis of these data, open coding was used for content analysis (Strauss & Corbin, 1990). The data obtained were examined based on the strategies they used and the use of procedural and conceptual knowledge while generating examples.

In order to determine the strategies students use while generating examples, the trial-and-error, transformation and analysis strategies, previously found in the literature, were benefitted and new findings that emerged during the study were taken into account. The coding scheme of example generation strategies is found in Table 3. Accordingly, a frequency table of example generation strategies used by students was created (Table 2).

Another category that is analyzed is the ability to generate examples according to how students use procedural and conceptual knowledge. During the analysis of procedural knowledge, it is usually asked to solve the problem and examined whether the result or the operation is correct (Rittle-Johnson & Schneider, 2015). However, the questions in this study were prepared mostly in a way that allows students to generate examples using conceptual knowledge. For this reason, conceptual knowledge has been analyzed more. In this process procedural knowledge that could come up with the conceptual knowledge was also analyzed in replies given by students.

Students are required to identify and explain the relevant concept in clinical interviews in order to determine their level of conceptual knowledge (Rittle-Johnson & Schneider, 2015). Accordingly, when one's conceptual knowledge is analyzed, the compatibility of this knowledge with the known and widely accepted concept definitions is examined. In addition, while the conceptual information is evaluated, examples given about the concept can be examined or the student can be referred to include different expressions (as an example of graphing a non-continuous function) (Yanık, 2014). In this context, after the application with the students, clinical interviews have been prepared in order to make a more detailed analysis.

The students were asked questions about the concepts found in the questionnaire, aiming to investigate the reasons and considerations of the responses; for example, "What did you think about this question?," "What do you understand when ….. is said?," " What does the function defined from Z to N mean?" In this context, they were asked to transfer their new ideas (if any) to practice forms belonging themselves. Thus, by evaluating the example generation process of students in the context of conceptual and procedural knowledge, the factors affecting example generation skills were investigated.

Analyzes of procedural and conceptual knowledge levels of students are planned for each application questionnaire as follows:

The analysis of the first six research questions will require the condition of function, so the following function definition has been used.

A subset of A x B cartesian set that provides "For each element in A, B only has one element such that A and B are two sets of elements so (a, b) ∈ f " this special condition, and a set of A to set B is called a function (Even, 1993).

In the first question of the research, it was expected that students would know what domain and codomain mean and how they should be used and to write a function rule by considering these sets. Domain of the function to be written must be integers, and codomain must be natural numbers. It will be said that there is a lack of concept knowledge of students who cannot give an appropriate example to this situation. Also, in this question, the operations that the students will take without considering domain and codomain of the function will be used when the answer to the question of whether the procedural knowledge is in the foreground.

In the second question, it is aimed to examine the graphic representation of conceptual information on the condition of being a function. In mathematics lessons this is usually explained by the vertical test. According to the vertical test, lines parallel to the y-axis are drawn and this graph does not specify a function if any of those lines pass through more than one point on the graph. However, it is not enough for this test to be known by students in terms of learning conceptual knowledge and it is expected to know why this test is needed. This test is usually learned and applied by the students as a procedural knowledge. Very few students can develop a conceptual knowledge by relating this to the definition of the function (Bayazit & Aksoy, 2013). It will be said that the conceptual knowledge of the students who can explain the vertical line test is sufficient.

In the third question, the students asked to give a function example related to daily life by considering the child-mother relation which many teachers use in mathematics lessons to explain the condition of function. The analyzes made here are more about whether the student can correctly determine domain, codomain and the rule of function that he or she has written.

In the fourth question of the research, it is requested to investigate whether the students can write the function rule with different variables other than the x independent and y dependent variables which are more commonly used in mathematics lessons. "m is the function of n" means n independent, m is dependent variable. In this case, it will be said that the conceptual knowledge level of the students who can determine the appropriate function rule is sufficient.

In the fifth question, it is aimed to reveal concept knowledge about solution set in first degree equations in one variable and in what case solution set can be empty set. In this question, the level of procedural knowledge can also be determined depending on the operations student has made in the solution of the equation. When analyzing the conceptual level of knowledge for this question, the following explanations will be taken into consideration:

In an equation of $ax + b = 0$, the x values are called the root of the equation. The set of roots is also called the solution set and it is indicated by "ÇK (initials of solution set)". If $a = 0$ and $b \neq 0$, the equation becomes $0.x + b = 0$. In this case, equality will never be true even if any real number value is written in place of variable x. The solution set becomes a null set and is expressed as ÇK = Ø (Maviş, Gül, Solaklıoğlu, Tarku, Bulut & Gökşen, 2017).

At the last question, students were asked to investigate limit and continuity at one point. This can be explained by both an algebraic expression or a graph. The following definitions shall be used for the analysis of student responses to the question:

The real number of L1 (L2) is called the limit of the $f$ function on the left (right) at point x=a if the values of f (x) approach to the reel number of L1 (L2) when x values approach to a from left (right), with less (greater) values than a in the function f defined as $f$ : R→ R or $f$ : R - {a} → R , y= f(x). This is expressed as $\lim_{x\to a^-} f(x)= L_1$ ($\lim_{x\to a^+} f(x)= L_2$). The limit of the function at this point is $L_1 = L_2$ =R and represented as $\lim_{x\to a} f(x)$= L if the left and right limits of the function f(x) at point x = a are equal to each other.

If $\lim_{x\to a} f(x)$= f(a) ∈ R for function $y = f(x)$, then this function is called "continuous at point x = a." (Keskin, 2016).

According to these definitions, the students who are able to make graphical drawings appropriate to the conditions required in the question will also have the right concept knowledge.

In order to ensure the reliability of the work, the voice recordings were listened to several times by the researchers while they were being transmitted to text. The data were examined and coded by both researchers independently. The consistency between the encoders was calculated by Cohen's Kappa coefficient and was found to be 83.75%. In cases where there was a difference between encoders, the researchers worked together on codes to arrive at a consensus. In order to increase the reliability of the study, the method of data analysis has been explained in detail and direct quotations from the interview data have been included. Abbreviations such as S-1, S-2, S-3, etc have been used instead of the real names of the students when the quotations are given.

## 3. Findings

This section includes findings related to the use of conceptual and procedural knowledge and the strategies students used while generating examples.

### 3.1. Example generation strategies used by students

In this section, the frequency of students' use of each example generation strategy were determined (Table 2), and the strategies they used were given examples.

Table 2: *Frequency of Use of Example Generation Strategies by Participating Students*

| Strategies | Frequency of Use | Ratio in All Strategies Used |
|---|---|---|
| Trial and Error | 22 | 22/26 (85 %) |
| Transformation | 4 | 4/26 (15 %) |
| Analysis | 0 | 0/26 (0 %) |

According to Table 2, all 22 students who participated in the interview used the trial and error strategy and 4 of them also used the transformation strategy. None of the students used the analysis strategy.

Table 3 shows the coding examples used when analyzing the data, according to example generation strategies and the subcategories related to them.

Table 3: *Coding scheme of example generation strategies*

| Categories | Coding Examples |
|---|---|
| Trial and Error | • Because it is a limit value and not continuous for saying i approached at one point from left to right on function. If we approach from the right and left and both have the same result, there is a limit and it is continuous. Actually I have given a function example which is continuous but you want discontinuous. If we thought of something like this, the function isn't continuous where it is "broken".<br>• m and n are variables. We will give a value to n and output of this value will be the elements in m. For example $f(x) = (m - n)x + n$ is the function of variables m and n. But i can write a function of x and y like this: $f(x) = x^2 + 5x + 1$. This one is a fuction according to x. So, if i write $n^3 + n + 5 = m$ it can be a function of m and n.<br>• I tried the function $f(x) = 4x + 5$, again it is not OK. Let's try $f(x) = -3x - 5$. |
| Transformation | • I wrote this function $f(x) = x^2 + 5x - 4$. But then f(0)=-4. To avoid a negative, I made a change $f(x) = x^4 + x^2 + 5$.<br>• When you think of integers, there are also negative ones. To avoid a negative, I wrote $x^2$ instead of x.<br>• Well, i randomly gave this example. Hmm this one is providing the condition of being a function. But, if i change the condition of the function like this, it is not a function, because a child cannot have two mothers.<br>• Well, I wrote a function like $f(m) = n$. I wonder if m is an output of n? No, it is not. Then, I changed the places of m and n. Now, it is OK $f(n) = m$. |
| Analysis | No relevant findings were found for the analysis strategy. |

3.1.1. Trial and error strategy

As found in previous studies (Antonini, 2006; Sağlam & Dost, 2015; Edwards & Alcock, 2010; Iannone et al., 2009), trial and error was the most used strategy by the participants (Table 2). Some participants used this strategy in the following way:

In the first question, S-22 generated random examples without regard to constraints such as the definition of function and number sets.

**Researcher**: How did you generate the function in this question?

**S- 22**: I wrote a function to natural numbers form integers. $f(x) = -3x + 5$ […] $x$'s must be natural numbers. It is […] $f(3) = -4$. -4 is not a natural number. It is an integer rather than a natural number. If it is positive, I think it will be OK. Let's say $f(x) = 4x + 5$. However, it is $f(-3) = -7$. Again it is negative.

In the last question of the research, student S-17 could not generate an appropriate example of the constraints on the question because he had lack of concept knowledge about limit and continuity. He used trial and error strategy and has concept image that the function is not continuous in the place where it is "broken".

**Researcher**: Can you explain your thoughts on this question?

**S- 17**: I approached at one point from left to right on function because it is a limit value and not continuous for saying. If we approach from the right and left and both have the same result, there is a limit and it is continuous. Actually I have given a function example which is continuous but you want discontinuous.

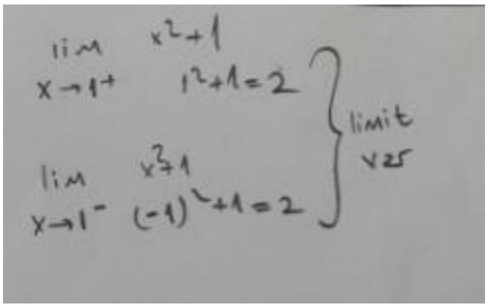



*Figure 1*. Student 17's answer to question 6

**Researcher**: Can you express it on the graphic?

**S- 17**: If we thought of something like this, the function isn't continuous where it is "broken".

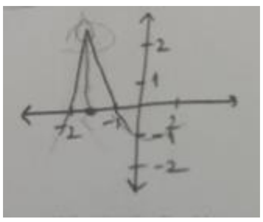


*Figure 2*. Student 17's the other answer to question 6

In the fourth question of the application, the students were asked to give a function example with different variables, rather than the common notations used in the functions.

Among the participants, the student S-12 could not generate a suitable example in the fourth question while doing random trials. However he later gave an example appropriate for the restrictions by using the function $f(x) = x^2 + 5x + 1$, depending on the variable x, which is the reference representation of the functions.

**Researcher**: Can we talk about the example you gave here? What comes to your mind when you hear m is n's function?

**S- 12**: m and n are variables. We will give n a value and it will be an element of m. […] For example, the function $f(x) = (m - n)x + n$ is the function of the variables m and n.

**Researcher**: How is the function expressed according to x and y variables?

**S- 12**: For example, the expression $f(x) = x^2 + 5x + 1$ is a function. A function according to x. […] Then, if I write $n^3 + n + 5 = m$, it is a function according to n.

3.1.2. Transformation strategy

In this strategy, the example that satisfies one or more of the requested properties is modified through some transformations until it turns into an example with all the requested characteristics. In this study, it was a less used strategy than trial and error (Table 2). Below are the answers that some participants gave by using the transformation strategy.

In the first question, student S-4 tried to write a function as a prototype, or the first example that came to his mind. But then he thought that the outputs were negative for some elements in the domain and did not provide the requested properties. By transforming the rule of function, the student wanted to give a function example defined from integers to natural numbers.

**Researcher**: Do your examples provide the desired characteristics from the question?

**S-4:** First, I got the function $f(x) = 2x$. When you say integers, there are also negative ones. […] I thought $x^2$ to eliminate the negative ones. Everything was positive now.


*Figure 3*. Student 4's answer to question 1

In the fourth question, Student S-17 could not distinguish between dependent/independent variables. But he tried to generate an appropriate example by changing the places of dependent and independent variables during the interview.

**Researcher**: Can you explain how you solved this question?

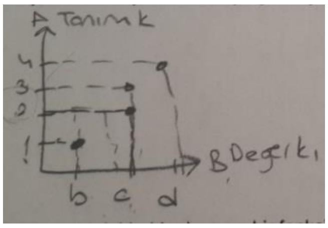
**S- 17**: Well.. First, I wrote $f(m) = n$ here. I wrote a function. That is, instead of x value, the value of $f(m)$ function is equal to n ….[thinking]… Then, will m be the output of n? No, it is not OK. If I write $f(n) = m$, it is OK.

**Researcher**: Can you write an appropriate algebraic rule for this?

**S- 17**: $(n) = n^2 + 2n = m$ .

In the second question of the application, while drawing a graphic that did not specify a function, the student S-2 gave an example based on the relation "mother-child" which is a reference and which is frequently used in her lessons. Although the first graphic that the student drew randomly specified a function (graphic on the left), the student changed the range without changing the domain and codomain of the function and generated a suitable example.

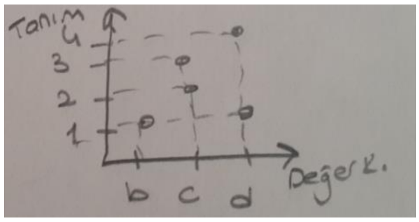
A (Domain)




B (Codomain)

*Figure 4.* Student 2's answer to question 2

A (Domain)




B (Codomain)

Figure 5. Student 2's the other answer to question 2

**Researcher:** Can you tell me the graphics you've drawn?

**S-2:** In this graphic [showing the graphic on the left], I considered domain (A) as children and codomain (B) as mothers. [thinking aloud] It must not be a function. Does every child have only one mother? Yes, it does. Well... It is a function.

**Researcher:** So, have you made a change on this chart?

**S-2:** Actually, I have not made much change. I have not changed domain and codomain. As you see here, [showing the graph on the right] I have changed the rule. This is not a function I have drawn because a child cannot have two mothers.

3.1.3. Analysis strategy

In this strategy, assuming the constructed object [exists], and possibly assuming that it satisfies other properties added in order to simplify or restrict the search ground, further properties are deduced, up to consequences that may evoke either a known object or a procedure to construct the requested one. It has been observed that students did not use the analysis strategy in the example generation activities included in this study. Analysis strategy has been seen very little in the example generation process at Iannone (2009) and Sağlam and Dost (2015) works with university students.

**3.2. Use of procedural and conceptual knowledge in example generation process**

In the process of generating examples for some mathematical concepts, it was seen that both the procedural and conceptual knowledge levels of the students are inadequate and for this reason they failed to generate examples. For example, student S-21 gave a function example for the first question and tried to determine whether he generated the example according to the desired conditions by performing some numerical operations on this function. However, as the student had a lack of knowledge about the function, the process of generating examples failed. It is also seen that there is a lack of procedural knowledge because the algebraic expression that the student wrote and the operations performed on it are incorrect.

**Researcher:** Can you tell me what did you think about this question?

**S-21:** For example, the function had an unknown. I assigned it. $f(x) = x^2 - 2 + 5 = 7$. We can also find the value x by doing some operations on the function. Hmm, is it okay with 1?

**Researcher:** Does your example provide these conditions? What is a function defined from Z to N?

**S-21:** Because the integers cover natural numbers. It's something like defined from big numbers to small numbers. [Thinking] Actually, the function I wrote is probably not very convenient. That is, it is not always 7. It may also be different numbers [... thinking] If I did not write 7, I would generate an appropriate example.

Concept images, which is a cognitive structure in which some features, methods and mental pictures of a concept take place (Tall & Vinner, 1981), have an important role in the example generation process. During the analysis of data for conceptual and procedural knowledge, it was seen that students also responded to the questions with the concept images in their mind. For example; the student S-20 tried to give a continuous function example by drawing a graph in the sixth question. For the concept of continuity, the student used a concept image as the graph's "flowing off." However, the student tries to find an answer to this question both algebraically and graphically it seems that he doesn't have the correct concept knowledge when the definitions for limit and continuity concepts are considered.

**Researcher**: Could you tell me your solution to this question?

**S-20**: In order to be continuous, the function must flow off. Is my function flowing off? [Thinking]

**Researcher**: So now is the function in the graph continuous?

**S-20**: The graph is flowing here [... showing a part on the graph]. But it is not continuous in -2 and 1. So this function is not continuous. Then what can I do? [Thinking].

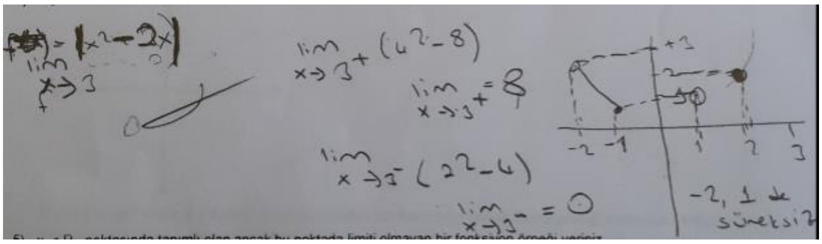
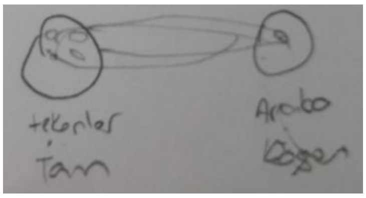



*Figure 6*. Student 20's answer to question 6

In the third question of the study, it was found that student S-11 had a concept image that is not mathematically compatible as "the elements of the codomain include the elements of the domain" for the definition of the function. Therefore, the student could not generate a suitable example for the constraints.

**Researcher:** Can you tell me what did you think about this question?

**S-11:** Since it comes from codomain to the domain, the car has all the wheels. I thought the elements of the codomain include all the other elements of the domain as function.



(Wheels Domain) (Car Codomain)

*Figure 7*. Student 11's answer to question 3

On the other hand, students who were able to use conceptual knowledge as well as procedural knowledge could give examples appropriate for the restrictions. For example, student S-1 has used the vertical line test for the condition of being a function in the second question. As the student explains what this test means, it can be said that she has the correct concept knowledge. So the student generated the desired example in this question.

**Researcher:** How did you solve this question?

**S-1:** I used vertical line test. Our teacher had given this example.

**Researcher:** What does the vertical line test mean?

**S-1:** It must have only one output in order to be a function. You see the vertical lines I draw here. They intercept the graph at more than one point. Actually, each child in the domain must have one mother. Here, it has more than one value. It is not a function.

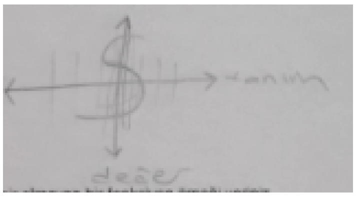


*Figure 8*. Student 1's answer to question 2

Similarly, student S-12 generated two appropriate examples, using the definitions of limit and continuity concepts, in the eighth question.

**Researcher:** Can you explain how did you solve the problem?

**S - 12**: The function must have a limit. That is, it must have a limit value at $x_0$ point. When we approach from right and left, it must have the same value but it must not be continuous there. [...] This function (showing the graph on the left) has a limit, but the function is discontinuous at the point $x_o$. Our teacher gave this example in the lesson.

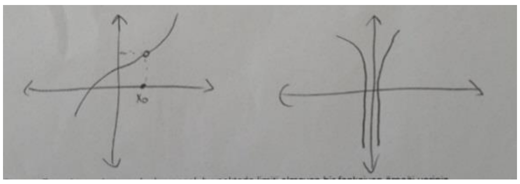


*Figure 9.* Student 12's answer to question 8

In the fifth question, the student S-15 generated an appropriate example of the first-degree equation in which the coefficient of the unknown on the left and right sides of the equation is equal but the remaining constants are not equal to each other (the answer to the question follows the general notion of equations with a solution set of empty sets). Thus, it can be said that the student had the right conceptual knowledge as well as procedural knowledge in this question.

**Researcher:** Could you explain your solution in this question?

**S-15:** I wrote an equation. For the solution set to be a null set, the right and left sides of the equation must not be equal to each other. Accordingly, the result of the equation I wrote is 0 = 1. So the solution set is a null set.

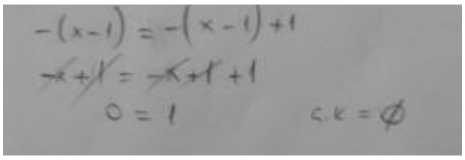


*Figure 10.* Student 15's answer to question 7

## 4. Conclusion and discussion

In this research, the strategies that high school students use in the process of generating examples in mathematics courses, and the effects of procedural and conceptual knowledge on example generation skills were investigated. According to the findings obtained from the study, the most used strategy in the process of example generation of high school students is trial and error and the least used strategy is the transformation strategy. As in the studies conducted by Edwards and Alcock (2010), Antonini (2006), Iannone et al. (2009) and Sağlam and Dost (2015), it was found that the most used strategy is trial and error.

When students used trial and error, they made random trials and could not determine whether their examples were appropriate or not because they did not have enough concept knowledge. However, during the interview, many students realized the mistakes they had made and made a second trial attempt. It was also found that encouraging students to use reference examples may be useful in generating appropriate examples. Arzarello, Ascari and Sabena (2011) also stated that the intervention of the teacher in the example generation activity has a critical importance. Students using the transformation strategy gave the prototypes as answers like in the trial and error strategy. However, when they found out that these examples were not appropriate to the required conditions in the question, they generated new examples by changing some features. In the study of Wagner et al. (2017), students made some changes on the examples they gave in order to provide the desired conditions in the example generation activity and used the transformation strategy. In addition, it was seen that example generation skills of students developed over time.

Example generation activities are an important pedagogical tool used to determine the level of competence of students in their understanding of related mathematical concepts. In this process, concept images of students (if any) can be determined. Everything in the mind of the student related to the concept represents concept images (Hershkowitz, 1990). The students answered some of the questions with concept images in their minds instead of concept definitions. Also, Tall and Vinner (1981) and Vinner (1991) achieved similar results in that students tend to use the concept image instead of using previous concept definitions in the process of generating a new concept. It was found that concept image was an incentive for the learner to comment on the question, but it was not enough by itself. We can say that students use concepts incorrectly because generally example generation activities are not included and specific examples (prototypes) are used in lessons. It can be said that all these factors negatively affect the students' ability to generate examples. In the study of Tezer and Cumhur (2016), the common conceptual mistakes made by the students during the problem solving in mathematics courses were determined by primary school teachers. However, it has been concluded that the constructivist approach, that provide to be designed environments and situations where mathematics to be taught and students product their own knowledge, can prevent these conceptual errors seen in students. In this context, designing classroom activities that enable students to generate examples helps them to understand the new mathematical concepts they encounter (Wagner et al., 2017).

Other factors considered when examining example generation activities are procedural and conceptual knowledge. We tried to determine the students' ability to generate examples based on this. Students are a good mirror for learning through memorizing; they skillfully reflect what comes to them, but they do not generate anything (Cobb, 1986). The students who have a lack of conceptual and procedural knowledge failed to relate concepts, make inferences and manipulate the information. For this reason, they gave mathematically incorrect or incomplete answers during the example generation process, in which conceptual knowledge and learning were required. The individual who learns a concept should be able to identify the concept with his/her own expressions and give examples related to the concept (Gagne, 1977). In this study, students with conceptual knowledge were able to generate examples appropriate for the restrains involved in the questions. In Baki (2004) and Soylu and Aydın's (2006) studies, it was seen that participants did not use conceptual and procedural knowledge in a balanced manner and procedural learning was at the foreground. Therefore, it can be said that students were unable to apply the concepts or descriptions they learned in class.

## 5. Recommendations

The high school students' lack of sufficient conceptual knowledge, the fact that procedural knowledge is at the foreground, and their having mathematically inappropriate concept images are the main causes that negatively affect the example generation process.

Example generation activities can be used for both evaluation and skill development. Whether or not students' conceptual learning has taken place and concept images (if any), can be determined through the activities used for the purpose of evaluation. After this assessment made by the mathematics teachers, the activities aiming at developing conceptual learning and example generation skills can be planned. In this context, teachers can include the following activities in their lessons: "Make up an example with some constraints," which is used to encourage students to use a general method rather than giving a random answer through trial and error; "Make up another or more like or unlike this," which informs teachers about the thoughts of students about the concept of similarity and draws attention to the different elements of possible variations; and "Make up counter-examples and non-examples" (Watson & Mason, 2005), which is used when a counter hypothesis or claim is needed in the context of a concept, procedure, or proof, and to show the boundaries of a concept.

In mathematics lessons and textbooks, there are often prototypes. These can lead to a student's limited perception of a concept, by drawing attention to only certain characteristics of the concept, preventing them from coping with complex problem situations. For this reason, both counter examples to clarify the distinctions between the concepts and to show that the results cannot be generalized (Wagner et al., 2017) and reference examples that can be used in different situations related to any topic, can help to eliminate and explain some mathematical unknowns (Michener, 1978) and should be included in courses and textbooks.

In addition to this, students should be encouraged to perform different types of research and activities for mathematical concepts outside the class hours.

It was also found in this study that having only procedural knowledge is not sufficient in problem solving activities. In order to achieve lasting learning in mathematics education, it is necessary for teachers to plan for conceptual comprehension as well as procedural knowledge. In this context, by including problem situations that students have not encountered before in mathematics teaching, their thinking at the conceptual level can be improved.

Finally, mathematics teacher education programs should prepare teacher candidates for the use of instructional examples.

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